SEVEN LESSONS IN THEORY OF INVERSIONS OF ORDER AND DETERMINANTS

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IN THEORY OF

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AND

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BY

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PREFACE

No student should enter upon the study of analytic geometry without an elementary knowledge of that part of the theory of determinants which treats of *elimination* and the *solution of simultaneous equations*. This will be the more readily conceded when it is shown that such knowledge may be gained in fewer than a dozen lessons. Seven introductory lessons in theory of inversions and determinants would scarcely suffice *per se* to make a lasting impression upon the student; but if the elementary principles thus learned are immediately applied to the solution of problems in analytic geometry the good gained from such a brief course should be very considerable. In whatever manner the student be introduced to a subject, it is, in most cases, only by constant recurrence and many varied illustrations that the value of a process is finally realized.

It is thought by the writer that the treatment of inversions as applied to determinants is new in some respects, and attention is invited to the *numbered* theorems and related matter which form a chain of reasoning leading up to Laplace's development.

UNIVERSITY OF MINNESOTA
April, 1902.

INVERSIONS AND DETERMINANTS

CHAPTER I

INVERSIONS OF ORDER

1. If several consecutive symbols of a sequence, as letters of the alphabet, or numerals, be written in a line in any order, then an inversion of natural order, or simply an **inversion**, occurs whenever any symbol follows another which it should *naturally* precede. In *bac*, 5647, 4321, *JKIHG*, 564213, there are respectively 1, 2, 6, 9, 12 inversions.

2. In considering any such line of symbols we shall number their positions to the right beginning at the left. Then the **order of position** of any symbol in the line is of the 1st, 2nd, 3d, degree according as it occupies the 1st, 2nd, 3d, position; and so on. The **order of a symbol,** and the degree of the order, is taken with reference to the natural sequence of the symbols in the line: it is the same as the order of the position it would occupy if the symbols were arranged in natural order in the positions of the line. It will be frequently convenient to refer to a symbol, or position, as being *even* or *odd*, meaning that the *order* of the symbol, or position, is of even or odd degree.

3. **Theorem I.** *In any line of symbols capable of arrangement in natural sequence, if any two symbols be interchanged the number of inversions in the line will be increased or decreased by an odd number.*

The interchange of any two symbols occupying consecutive positions obviously causes a loss or gain of a single inversion.

If there are m symbols intervening between the two to be interchanged, then the transfer of each of the two to the position of the other obviously causes a loss or gain of one inversion with each intermediate; moreover there is also a loss or gain of one inversion between the two symbols interchanged. Therefore there are $2m+1$, an odd number, of losses and gains *together*. But the change in the number of inversions is the *difference* between losses and gains, and if the sum of two integers be odd their difference is odd. Therefore the change in the number of inversions is odd in any case.

4. **A complex symbol** can be formed by uniting into a single symbol of two simple parts any two symbols chosen from as many different sets of sequences of the kind we have been considering. A *triply* complex symbol may be formed by choosing from three sets of sequences; and so on. As illustrations, (11), $\overset{h}{\underset{x}{\cdot}}$, b_3, A_4, Q^a, a^2, B^c, are doubly, and $A\overset{b}{\underset{y}{-}}$, $m^{5/6}$, ${}^8X_a$, triply complex. We shall confine our remarks to the first kind.

5. Definition. The **order of a complex symbol** is the sum of the orders of its simple parts.

6. **Theorem II.** *If to any number of consecutive letters taken in any order, the same number of consecutive numerical suffixes be attached, one suffix to each letter, then upon writing these complex symbols in line in any order at pleasure, the total number of inversions among both letters and suffixes will be either always odd or always even.*

In any *one* interchange of two symbols the number of inversions among either letters or suffixes is changed by an odd number (Theorem I); and the sum or difference of two odd numbers is an even number. Hence the total number of inversions after one interchange remains either odd or even as it was. But by successive interchanges two at a time the symbols can be brought into any prescribed order one at a time. The truth of the theorem is apparent.

Example. The number of inversions in $a_1b_2c_3d_4$, $d_4c_3a_1b_2$, $d_4c_3b_2a_1$, or any other permutation of the group, is respectively 0, 10, 12, even.

Scholium. The greatest possible number of inversions in such groups is $n(n-1)$, where n is the number of complex symbols.

7. **Theorem III.** *In any line of doubly complex symbols, whose letters and suffixes are the members of corresponding sequences, the total number of inversions due to the presence of any specified complex symbol, H_x, is even or odd according as the order of that symbol is even or odd.*

If the order of H_x is even, then H and x are both even or both odd; consequently there must be present an *even* number of letters and suffixes together which are of lower orders than H, x respectively. If H_x is odd, then H, x are one odd the other even, and there must be present an *odd* number of letters and suffixes together of lower orders than H, x respectively. Therefore the number of inversions due to H_x in the *first* position of the line is even or odd according as H_x is even or odd, since the only inversions due to H_x in that position are with letters and suffixes of lower orders than H, x respectively. But the number of inversions due to H_x is always even or always odd, independent of the arrangement of the line, since in any one interchange, and consequently in any succession of interchanges, it is impossible for H_x to gain or lose an odd number of inversions with any other doubly complex symbol. The theorem follows.

Cor. *The number of inversions due to the symbol H_x is even or odd according as $(-1)^r$ is $+$ or $-$; r being the order of H_x.*

Cor. *If there be a line of simple symbols arranged in any order, then the number of inversions in the line due to the presence of any specified symbol of the sequence will be even or odd according as the sum of the orders of the specified symbol and its position is even or odd.*

For a line of doubly complex symbols may be written with its letters in natural order and its order of suffixes corresponding to the order of symbols in the given line. Then the order of position of any suffix is the same as the order of its literal partner, and the number of inversions in the given line is the same as the number in the written line. The corollary follows from this and Theorem III.

8. Definition. The **order of a group,** of two or more complex symbols, is the sum of the orders of the constituents of the group.

9. **Theorem IV.** *The number of inversions in any line of doubly complex symbols, due to the presence of any specified two or more such symbols, is even or odd according as* $(-1)^{s+i}$ *is* $+$ *or* $-$; *s being the order of the specified group and i its number of inversions.*

Let S be the sum of the numbers of inversions respectively due to each symbol of the specified group considered separately. Then S includes each inversion occuring among the specified symbols *twice*, and each inversion between a symbol within the group and another without the group but once. Therefore $S-i$ is the number of inversions due to the specified group in the line. But S and s are both even or both odd (Theorem III and properties of numbers). Therefore the number of inversions due to the group is even or odd according as $S-i$, and consequently as $s-i$, is even or odd; that is according as $(-1)^{s-i}$, and therefore as $(-1)^{s+i}$, is $+$ or $-$.

Cor. *Theorem III is a special case of Theorem IV, since then* $s=r$ *and* $i=0$.

Examples.

1. Is the number of inversions in $a_2\,c_1\,f_7\,g_6\,d_3\,e_4\,b_5$, due to the presence of $a_2\,g_6\,d_3\,b_5$, even or odd? For ease of enumeration represent the groups by $\frac{1\ 3\ 6\ 7\ 4\ 5\ 2}{2\ 1\ 7\ 6\ 3\ 4\ 5}$ and $\frac{1\ 7\ 4\ 2}{2\ 6\ 3\ 5}$. In the latter $(s-i)=30-5=25$, and $(-1)^{25}$ is $-$. Hence there should be an *odd* number of inversions due to $\frac{1\ 7\ 4\ 2}{2\ 6\ 3\ 5}$; and by actual count there are 17 in $\frac{1\ 3\ 6\ 7\ 4\ 5\ 2}{2\ 1\ 7\ 6\ 3\ 4\ 5}$ and 2 in $\frac{3\ 6\ 5}{1\ 7\ 4}$, leaving 15, an *odd* number, due to $\frac{1\ 7\ 4\ 2}{2\ 6\ 3\ 5}$.

2. In Ex. 1 $s-i=24$ for $\frac{3\ 6\ 5}{1\ 7\ 4}$; and accordingly there are 12 inversions due to $\frac{3\ 6\ 5}{1\ 7\ 4}$.

3. In $d_1\,a_2\,c_4\,b_3\,e_5$ there are 6 inversions due to the presence of $c_4\,e_5\,d_1$ for which $s-i=16$ an *even* number.