ON MULTIPLE ALGEBRA. AN ADDRESS BEFORE THE SECTION OF MATHEMATICS AND ASTRONOMY OF THE AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE AT THE BUFFALO MEETING, AUGUST, 1886

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On multiple algebra. An address before the Section of Mathematics and Astronomy of the American Association for the Advancement of Science at the Buffalo meeting, August, 1886 by J. Willard Gibbs

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BY

J. WILLARD GIBBS,

VICE-PRESIDENT.

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ADDRESS

BY

J. WILLARD GIBBS,

VICE PRESIDENT, SECTION A, MATHEMATICS AND ASTRONOMY.

MULTIPLE ALGEBRA.

It has been said that "the human mind has never invented a labor-saving machine equal to algebra."[1] If this be true, it is but natural and proper that an age like our own, characterized by the multiplication of labor-saving machinery, should be distinguished by an unexampled development of this most refined and most beautiful of machines. That such has been the case, none will question. The improvement has been in every part. Even to enumerate the principal lines of advance would be a task for any one; for me an impossibility. But if we should ask, in what direction the advance has been made, which is to characterize the development of algebra in our day, we may, I think, point to that broadening of its field and methods, which gives us *multiple algebra*.

Of the importance of this change in the conception of the office of algebra, it is hardly necessary to speak: that it is really characteristic of our time will be most evident if we go back some two- or threescore years, to the time when the seeds were sown which are now yielding so abundant a harvest. The failure of Möbius, Hamilton, Grassmann, Saint-Venant to make an immediate impression upon the course of mathematical thought in any way commensurate with the importance of their discoveries is the most conspicuous evidence that the times were not ripe for the methods which they sought to introduce. A satisfactory theory of the imaginary quantities of ordinary algebra, which is essentially a simple case of multiple algebra, with difficulty obtained recogni-

[1] *The Nation*, Vol. XXXIII, p. 237.

tion in the first third of this century. We must observe that this *double algebra*, as it has been called, was not sought for or invented; —it forced itself, unbidden, upon the attention of mathematicians, and with its rules already formed.

But the idea of double algebra, once received, although as it were unwillingly, must have suggested to many minds, more or less distinctly, the possibility of other multiple algebras, of higher orders, possessing interesting or useful properties.

The application of double algebra to the geometry of the plane suggested not unnaturally to Hamilton the idea of a triple algebra which should be capable of a similar application to the geometry of three dimensions. He was unable to find a satisfactory triple algebra, but discovered at length a quadruple algebra, *quaternions*, which answered his purpose, thus satisfying, as he says in one of his letters, an intellectual want which had haunted him at least fifteen years. So confident was he of the value of this algebra, that the same hour he obtained permission to lay his discovery before the Royal Irish Academy, which he did on November 13, 1843.[1] This system of multiple algebra is far better known than any other, except the ordinary double algebra of imaginary quantities,— far too well known to require any especial notice at my hands. All that here requires our attention is the close historical connection between the imaginaries of ordinary algebra and Hamilton's system, a fact emphasized by Hamilton himself and most writers on quaternions. It was quite otherwise with Möbius and Grassmann.

The point of departure of the *Barycentrischer Calcul* of Möbius, published in 1827,— a work of which Clebsch has said that it can never be admired enough,[2]— is the use of equations in which the terms consist of letters representing points with numerical coëfficients, to express barycentric relations between the points. Thus, that the point S is the centre of gravity of weights, a, b, c, d, placed at the points A, B, C, D, respectively, is expressed by the equation

$$(a + b + c + d)S = aA + bB + cC + dD.$$

An equation of the more general form

$$aA + bB + cC + \text{etc.}, = pP + qQ + rR + \text{etc.}$$

[1] *Phil. Mag.* (3), Vol. XXV, p. 490; *North British Review*, Vol. XLV (1866), p. 57.
[2] See his eulogy on Plücker, p. 14, *Gött. Abhandl.*, Vol. XVI.

signifies that the weights a, b, c, etc., at the points A, B, C, etc., have the same sum and the same centre of gravity as the weights p, q, r, etc., at the points P, Q, R, etc., or, in other words, that the former are barycentrically equivalent to the latter. Such equations, of which each represents four ordinary equations, may evidently be multiplied or divided by scalars,[1] may be added or subtracted, and may have their terms arranged and transposed, exactly like the ordinary equations of algebra. It follows that the elimination of letters representing points from equations of this kind is performed by the rules of ordinary algebra. This is evidently the beginning of a quadruple algebra, and is identical, as far as it goes, with Grassmann's marvellous geometrical algebra.

In the same work we find, also, for the first time, so far as I am aware, the distinction of positive and negative consistently carried out on the designation of segments of lines, of triangles and of tetrahedra, viz., that a change in place of two letters, in such expressions as AB, ABC, $ABCD$, is equivalent to prefixing the negative sign. It is impossible to overestimate the importance of this step, which gives to designations of this kind the generality and precision of algebra.

Moreover, if A, B, C are three points in the same straight line, and D any point outside of that line, the author observes that we have

$$AB + BC + CA = 0,$$

and, also, with D prefixed,

$$DAB + DBC + DCA = 0.$$

Again, if A, B, C, D are four points in the same plane, and E any point outside of that plane, we have

$$ABC - BCD + CDA - DAB = 0,$$

and also, with E prefixed,

$$EABC - EBCD + ECDA - EDAB = 0.$$

The similarity to multiplication in the derivation of these formulæ cannot have escaped the author's notice. Yet he does not seem to have been able to generalize these processes. It was re-

[1] I use this term in Hamilton's sense, to denote the ordinary positive and negative quantities of algebra. It may, however, be observed that in most cases in which I shall have occasion to use it, the proposition would hold without exclusion of imaginary quantities,— that this exclusion is generally for simplicity and not from necessity.

served for the genius of Grassmann to see that AB might be regarded as the product of A and B, DAB as the product of D and AB, and $EABC$ as the product of E and ABC. That Möbius could not make this step was evidently due to the fact that he had not the conception of the addition of other multiple quantities than such as may be represented by masses situated at points. Even the addition of vectors (*i. e.*, the fact that the composition of directed lines could be treated as an addition,) seems to have been unknown to him at this time, although he subsequently discovered it, and used it in his *Mechanik des Himmels*, which was published in 1843. This addition of vectors, or *geometrical addition*, seems to have occurred independently to many persons.

Seventeen years after the *Barycentrischer Calcul*, in 1844, the year in which Hamilton's first papers on quaternions appeared in print, Grassmann published his *Lineale Ausdehnungslehre*, in which he developed the idea and the properties of the *external* or *combinatorial product*, a conception which is perhaps to be regarded as the greatest monument of the author's genius. This volume was to have been followed by another, of the nature of which some intimation was given in the preface and in the work itself. We are especially told that the *internal product*,[1] which for vectors is identical except in sign with the scalar part of Hamilton's product (just as Grassman's external product of two vectors is practically identical with the vector part of Hamilton's product), and the *open product*,[2] which in the language of to-day would be called a matrix, were to be treated in the second volume. But both the internal product of vectors and the open product are clearly defined, and their fundamental properties indicated, in this first volume.

This remarkable work remained unnoticed for more than twenty years, a fact which was doubtless due in part to the very abstract and philosophical manner in which the subject was presented. In consequence of this neglect, the author changed his plan, and instead of a supplementary volume, published in 1862 a single volume entitled *Ausdehnungslehre*, in which were treated, in an entirely different style, the same topics as in the first volume, as well as those which he had reserved for the second.

Deferring for the moment the discussion of these topics in order to follow the course of events, we find in the year following the

[1] See the preface. [2] See § 172.

first *Ausdehnungslehre* a remarkable memoir of Saint-Venant[1], in which are clearly described the addition both of vectors and of oriented areas, the differentiation of these with respect to a scalar quantity, and a multiplication of two vectors and of a vector and an oriented area. These multiplications, called by the author *geometrical*, are entirely identical with Grassmann's external multiplication of the same quantities.

It is a striking fact in the history of the subject, that the short period of less than two years was marked by the appearance of well-developed and valuable systems of multiple algebra by British, German, and French authors, working apparently entirely independently of one another. No system of multiple algebra had appeared before, so far as I know, except such as were confined to additive processes with multiplication by scalars, or related to the ordinary double algebra of imaginary quantities. But the appearance of a single one of these systems would have been sufficient to mark an epoch, perhaps the most important epoch in the history of the subject.

In 1853 and 1854, Cauchy published several memoirs on what he called *clefs algébriques*.[2] These were units subject generally to combinatorial multiplication. His principal application was to the theory of elimination. In this application, as in the law of multiplication, he had been anticipated by Grassmann.

We come next to Cayley's celebrated *Memoir on the Theory of Matrices*[3] in 1858, of which Sylvester has said that it seems to him to have ushered in the reign of Algebra the Second.[4] I quote this dictum of a master as showing his opinion of the importance of the subject and of the memoir. But the foundations of the theory of matrices, regarded as multiple quantities, seem to me to have been already laid in the *Ausdehnungslehre* of 1844. To Grassmann's treatment of this subject we shall recur later.

After the *Ausdehnungslehre* of 1862, already mentioned, we come to Hankel's *Vorlesungen über die complexen Zahlen*, 1867. Under this title the author treats of the imaginary quantities of ordinary algebra, of what he calls *alternirende Zahlen*, and of quaternions. These alternate numbers, like Cauchy's *clefs*, are quantities subject to Grassmann's law of combinatorial multiplication. This treatise, published twenty-three years after the

[1] *C. R.* Vol. XXI, p. 620. [2] *C. R.* Vols. XXXVI, ff. [3] *Phil. Trans.* Vol. CXLVIII. [4] *Amer. Journ. Math.* Vol. VI, p. 271.

first *Ausdehnungslehre*, marks the first impression which we can discover of Grassmann's ideas upon the course of mathematical thought. The transcendent importance of these ideas was fully appreciated by the author, whose very able work seems to have had considerable influence in calling the attention of mathematicians to the subject.

In 1870, Professor Benjamin Peirce published his *Linear Associative Algebra*, subsequently developed and enriched by his son, Professor C. S. Peirce. The fact that the edition was lithographed seems to indicate that even at this late date a work of this kind could only be regarded as addressed to a limited number of readers. But the increasing interest in such subjects is shown by the republication of this memoir in 1881,[1] as by that of the first *Ausdehnunsglehre* in 1878.

The article on quaternions which has just appeared in the *Encyclopædia Brittanica* mentions twelve treatises, including second editions and translations, besides the original treatises of Hamilton. That all the twelve are later than 1861 and all but two later than 1872 shows the rapid increase of interest in this subject in the last years.

Finally, we arrive at the *Lectures on the Principles of Universal Algebra* by the distinguished foreigner whose sojourn among us has given such an impulse to mathematical study in this country. The publication of these lectures, commenced in 1884 in the *American Journal of Mathematics*, has not as yet been completed,—a want but imperfectly supplied by the author's somewhat desultory publication of many remarkable papers on the same subject (which might be more definitely expressed as the algebra of matrices) in various foreign journals.

It is not an accident that this century has seen the rise of multiple algebra. The course of the development of ideas in algebra and in geometry, although in the main independent of any aid from this source, has nevertheless to a very large extent been of a character which can only find its natural expression in multiple algebra.

Our Modern Higher Algebra is especially occupied with the theory of linear transformations. Now what are the first notions which we meet in this theory? We have a set of n variables, say

[1] *Amer. Journ. Math.*, Vol. IV.